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ON OPTIMALITY AND DUALITY FOR MULTIOBJECTIVE MATHEMATICAL PROGRAMMING PROBLEMS WITH EQUILIBRIUM CONSTRAINTS USING GENERALIZED CONVEXITY

Kunwar V. K. Singh[a], Jitendra Kumar Maurya[a],
Yogendra Pandey[b] and S. K. Mishra[a,*]

a. Department of Mathematics, Banaras Hindu University, Varanasi-221005, India.
b. Department of Mathematics, R. N. College (L.N. Mithila University, Darbhanga), India.
*Corresponding author bhu.skmishra@gmail.com

Abstract:

In this paper, we consider multiobjective mathematical programming problems with equilibrium constraints (MMPEC). We extend the concept of type I and pseudoquasi-type I functions for multiobjective mathematical programming problems with equilibrium constraints. We establish necessary and sufficient optimality conditions for multiobjective mathematical programming problems with equilibrium constraints under assumptions of generalized convexity. Further, we propose Wolfe type dual (WDMMPEC) and Mond-Weir type dual (MWDMMPEC). We establish weak duality and strong duality results under assumptions of generalized convexity.

Keywords: Mathematical programming problems with equilibrium constraints, Optimality conditions; Type I functions; Pseudoquasi-type I function; Efficient solutions.

Introduction

Multiobjective mathematical programming problems with equilibrium constraints is a constrained optimization problem whose constraints include complementarity constraints defined as follows:

$$\text{(MMPEC)} \qquad \min(f_1(z),\ f_2(z), \dots, f_l(z))$$

$$\text{subject to} \quad g(z) \leqq 0\ , \quad h(z) = 0,$$

$$G(z) \geqq 0, \quad H(z) \geqq 0,$$

$$G(z)^t H(z) = 0,$$

where $f: \mathbb{R}^n \longrightarrow \mathbb{R}^l,\ g: \mathbb{R}^n \longrightarrow \mathbb{R}^p,\ h: \mathbb{R}^n \longrightarrow \mathbb{R}^q, G: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $H: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are continuously differentiable on $\mathbb{R}^n$ and $G(z)^t$ indicates the transpose of $G(z)$.

The concept of mathematical programming problems with equilibrium constraints (MPEC) is coined by Harker and Pang [1] in 1988. Optimization problems with equilibrium constraints arise frequently in various real world problems e.g., in chemical process engineering, hydroeconomic river basin model, capacity enhancement in traffic, dynamic pricing in telecommunication networks and multilevel games (see, [2-3]). Mathematical programming problems with equilibrium constraints form a relatively new and interesting subclass of nonlinear programming problems. Chemical process industries require the solution of the nonlinear problems as a part of current process synthesis, design optimization and control activities.

Luo *et al.* [4] presented a comprehensive study of mathematical programs with equilibrium constraints. Fukushima and Pang [5] studied some feasibility conditions in mathematical programs with equilibrium constraints. Outrata [6] established necessary optimality conditions for a class of mathematical programs with equilibrium constraints. Scheel and Scholtes [7] studied mathematical programs with complementarity constraints and introduced several stationary point concepts. Ye [8] considered mathematical programs with equilibrium constraints and introduced various stationary conditions and established that it is sufficient for local or global optimal under quasi and pseudo convexity assumptions and obtained new constraint qualifications. Further, Flegel and Kanzow [9] introduced a new Abadie-type constraint qualification and a new Slater-type constraint qualification for mathematical programs with equilibrium constraints.

The concept of invexity was introduced by Hanson [10] as a generalization of convexity, then Kaul and Kaur [11] discussed the interrelations between η-convex, η-quasiconvex and η-pseudoconvex functions. Hanson and Mond [12] introduced two new class of functions called type I and type II functions, which are necessary and sufficient conditions for optimality in primal and dual problems respectively. Rueda and Hanson [13] defined pseudo-type I and quasi-type I functions and obtained sufficient optimality criteria for a nonlinear programming problems involving these functions. Kaul *et al.* [14] defined quasipseudo-type I and pseudoquasi-type I functions and obtained necessary and sufficient optimality criteria for a nonlinear programming problems involving these functions.

To the best of our knowledge, there are only few papers on multiobjective mathematical programming problems with equilibrium constraints (MMPEC); (see, Bao *et al.* [15], Mordukhovich [16] and Pandey and Mishra [17]). Mishra and Jaiswal [18] defined semi-infinite mathematical programming problems with equilibrium constraints (SIMPEC) and established optimality conditions and duality for the (SIMPEC). Recently, Pandey and Mishra [17] defined multiobjective semi-infinite mathematical programming problems with equilibrium constraints and defined the concept of Mordukhovich stationary point for the nonsmooth semi-infinite mathematical programming problems with equilibrium constraints in terms of the Clarke sub differential.

In this paper, we extend the concept of Mordukhovich stationary point (M-stationary point) and No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) for multiobjective mathematical programming problems with equilibrium constraints. In Sect. 2, we give some preliminary definitions. In Sect. 3, we derive necessary and sufficient conditions for multiobjective mathematical programming problems with equilibrium constraints. In Sect. 4, we propose Wolfe type dual and

Mond-Weir type dual model. Further, we establish weak and strong duality results for multiobjective mathematical programming problems with equilibrium constraints.

2. Preliminaries

The following convention for equalities and inequalities will be used. If $x, y \in \mathbb{R}^n$, then

$$x = y \text{ iff } x_i = y_i,\ i = 1, \dots, n;$$

$$x \leqq y \text{ iff } x_i \leqq y_i,\ i = 1, \dots, n;$$

$$x \leq y \text{ iff } x \leqq y \text{ and } x \neq y\,;$$

$$x < y \text{ iff } x_i < y_i,\ i = 1, \dots, n.$$

Throughout this paper P denote the set of feasible solution of the (MMPEC). Given a feasible vector z^* for the (MMPEC), we define the following index sets:

$$I_g = \left\{ i : g\left(z^*\right) = 0 \right\},$$

$$\alpha = \alpha(z^*) = \left\{ i : G_i(z^*) = 0, \quad H_i(z^*) > 0 \right\},$$

$$\beta = \beta(z^*) = \left\{ i : G_i(z^*) = 0, \quad H_i(z^*) = 0 \right\},$$

$$\gamma = \gamma(z^*) = \left\{ i : G_i(z^*) > 0, \quad H_i(z^*) = 0 \right\}.$$

The set β is known as degenerate set. If β is empty, the vector z^* is said to satisfy the strict complementarity condition. For further study, we define the following index sets:

$$J^+ = \left\{ i : \lambda_i^h > 0 \right\},\ J^- = \left\{ i : \lambda_i^h < 0 \right\},$$

$$\beta^+ = \left\{ i \in \beta : \lambda_i^G > 0,\ \lambda_i^H > 0 \right\},$$

$$\beta_G^+ = \left\{ i \in \beta : \lambda_i^G = 0,\ \lambda_i^H > 0 \right\},\ \beta_G^- = \left\{ i \in \beta : \lambda_i^G = 0,\ \lambda_i^H < 0 \right\},$$

$$\beta_H^+ = \left\{ i \in \beta : \lambda_i^H = 0,\ \lambda_i^G > 0 \right\},\ \beta_H^- = \left\{ i \in \beta : \lambda_i^H = 0,\ \lambda_i^G < 0 \right\},$$

$$\alpha^+ = \left\{ i \in \alpha : \lambda_i^G > 0 \right\},\ \alpha^- = \left\{ i \in \alpha : \lambda_i^G < 0 \right\},$$

$$\gamma^+ = \left\{ i \in \gamma : \lambda_i^H > 0 \right\},\ \gamma^- = \left\{ i \in \gamma : \lambda_i^H < 0 \right\}.$$

Definition 2.2. A feasible point z^* is said to be a local efficient solution of the (MMPEC) if there exists a neighbourhood of z^* such that for any $z \in \mathbb{R}^n \cap U$ the following cannot hold

$$f_i(z) \leqq f_i(z^*), \quad \forall\, i = 1,\dots,l,\, i \neq j,$$

$$f_j(z) < f_j(z^*), \quad \textit{for some } j.$$

Definition 2.3. A feasible point z^* is said to be an efficient solution of the (MMPEC) if for any $z \in \mathbb{R}^n$ the following cannot hold

$$f_i(z) \leqq f_i(z^*), \quad \forall\, i = 1,\dots,l,\, i \neq j,$$

$$f_j(z) < f_j(z^*), \quad \textit{for some } j.$$

Definition 2.4. A feasible point z^* is said to be a weak efficient solution of the (MMPEC) if for any $z \in \mathbb{R}^n$ the following cannot hold

$$f_i(z) < f_i(z^*), \quad \forall\, i = 1,\dots,l.$$

3. Optimality conditions for (MMPEC)

Following definitions are extension of Kaul *et al.* [14] for multiobjective mathematical programming problems with equilibrium constraints.

Definition 3.1. (Type I) Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^l$, $g: \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $h: \mathbb{R}^n \longrightarrow \mathbb{R}^q$, $G: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $H: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are continuously differentiable on $\mathbb{R}^n$and defined for the (MMPEC), then (f,g,h,G,H) is said to be type I with respect to η at $z^* \in P$ if there exists a vector function $\eta(z,z^*)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$such that, for all $z \in P$.

$$f(z) - f(z^*) \geqq \left[\nabla f(z^*)\right]^t \eta(z,z^*), \tag{3.1}$$

$$-g(z^*) \geqq \left[\nabla g(z^*)\right]^t \eta(z,z^*),$$

$$-h(z^*) = \left[\nabla h(z^*)\right]^t \eta(z,z^*),$$

$$-G(z^*) \leqq \left[\nabla G(z^*)\right]^t \eta(z,z^*),$$

$$-H(z^*) \leqq \left[\nabla H(z^*)\right]^t \eta(z,z^*).$$

If in the above definition the inequality (3.1) is strict, then we say that (f,g,h,G,H) is semi strictly-type I at z^*.

Definition 3.2. (Pseudoquasi-type I) Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^l$, $g: \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $h: \mathbb{R}^n \longrightarrow \mathbb{R}^q$, $G: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $H: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, are continuously differentiable on $\mathbb{R}^n$ and defined for the (MMPEC), then (f,g,h,G,H) is said to be pseudoquasi-type I with respect to η at $z^* \in P$ if there exists a vector function $\eta(z,z^*)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for all $z \in P$.

$$\left[\nabla f(z^*)\right]^t \eta(z,z^*) \geqq 0 \Rightarrow f(z) \geqq f(z^*),$$

$$-g(z^*) \leqq 0 \Rightarrow \left[\nabla g(z^*)\right]^t \eta(z,z^*) \leqq 0,$$

$$-h(z^*) \leqq 0 \Rightarrow \left[\nabla h(z^*)\right]^t \eta(z,z^*) \leqq 0,$$

$$-G(z^*) \leqq 0 \Rightarrow \left[\nabla G(z^*)\right]^t \eta(z,z^*) \geqq 0,$$

$$-H(z^*) \leqq 0 \Rightarrow \left[\nabla H(z^*)\right]^t \eta(z,z^*) \geqq 0.$$

Example 3.3. Consider the following MMPEC in $\mathbb{R}^2$.

$$\min \quad f(z_1,z_2) = (z_1^2, \quad z_1^2 + z_2)$$

$$\text{subject to} \quad G(z_1,z_2) = z_1 \geqq 0,$$

$$H(z_1,z_2) = z_2 \geqq 0,$$

$$G(z_1,z_2)H(z_1,z_2) = z_1 z_2 = 0, \quad \forall\, z_1,z_2 \in \mathbb{R}.$$

Let $f_1(z_1,z_2) = z_1^2$ and $f_2(z_1,z_2) = z_1^2 + z_2$. The feasible region of MMPEC is $P = \{(z_1,z_2) \in \mathbb{R}^2$ such that either $z_1 = 0$ and $z_2 \geqq 0$ or $z_2 = 0$ and $z_1 \geqq 0\}$.

If we take any point in feasible region and $\eta(z,z^*) = z - z^*$, here $z = (z_1,z_2)$ then, above example is type I as well as pseudoquasi-type I.

Following Definitions are extension of Definition 2.6 and 2.10 of Ye [8] for multiobjective mathematical programming problems with equilibrium constraints.

Definition 2.1. (NNAMCQ) Let z^* be a feasible point for the (MMPEC), where all functions are continuously differentiable at z^*. We say that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at z^* if there is no non zero vector $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2m}$ such that

$$\sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} \left[\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)\right] = 0,$$

$$\lambda_{I_g}^g \geqq 0, \quad \lambda_\gamma^G = 0, \lambda_\alpha^H = 0, \forall i \in \beta \text{ either } \lambda_i^g > 0, \lambda_i^G > 0, \text{ or } \lambda_i^H \lambda_i^G = 0.$$

Following definition is extension of Ye [8] for the multiobjective mathematical programming problems with equilibrium constraints.

Definition 3.4. (M-stationary point). A feasible point z^* of the (MMPEC) is called Mordukhovich stationary point if there exist $\tau = (\tau_1, \ldots, \tau_l) \geqq 0$ and $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2m}$, such that

$$\sum_{i=1}^{l} \tau_i \nabla f_i(z^*) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} \left[\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*) \right] = 0,$$

$$\lambda_{I_g}^g \geqq 0, \quad \lambda_\gamma^G = 0, \lambda_\alpha^H = 0, \forall i \in \beta \text{ either } \lambda_i^g > 0, \lambda_i^G > 0, \text{ or } \lambda_i^H \lambda_i^G = 0.$$

In this section, we derive optimality condition for multiobjective mathematical programming problems with equilibrium constraints and in Theorem 3.5, we extend the Theorem 2.1 of Ye [8].

Theorem 3.5. Let z^* be a local efficient solution for the (MMPEC) where all functions are continuously differentiable at z^*, then there exist $\tau = (\tau_1, \ldots, \tau_l) \geqq 0$ and $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2m}$, not all zero, such that

$$\sum_{i=1}^{l} \tau_i \nabla f_i(z^*) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} \left[\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*) \right] = 0, \quad (3.2)$$

$$\lambda_{I_g}^g \geqq 0, \quad \lambda_\gamma^G = 0, \lambda_\alpha^H = 0, \forall i \in \beta \text{ either } \lambda_i^g > 0, \lambda_i^G > 0, \text{ or } \lambda_i^H \lambda_i^G = 0.$$

Proof.

We reformulate the (MMPEC) by introducing slack variables in the following equivalent form

$$\text{MMPEC} \quad \min(f_1(z),\ f_2(z), \ldots, f_l(z))$$

$$\text{subject to} \quad g(z) \leqq 0, \quad h(z) = 0,$$

$$G(z) - x = 0, \quad H(z) - y = 0,$$

where $\Omega = \{(x, y) \in \mathbb{R}^{2m} : x \geqq 0, y \geqq 0,\ x^t y = 0\}$.

This is an optimization problem with equalities, inequalities and non-convex abstract constraint $(x, y) \in \Omega$ with $(x^*, y^*, z^*) = (G(x^*), H(y^*), z^*)$ as a local efficient solution. We conclude that there exist $\tau = (\tau_1, \ldots, \tau_l) \geqq 0$ and $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2m}$, not all zero and $(\xi, \gamma) \in N_\Omega(x^*, y^*)$, then the limiting normal cone of Ω at the point (x^*, y^*) such that

$$\sum_{i=1}^{l} \tau_i \begin{pmatrix} 0 \\ 0 \\ \nabla f_i(z^*) \end{pmatrix} + \sum_{i\in I_g} \lambda_i^g \begin{pmatrix} 0 \\ 0 \\ \nabla g_i(z^*) \end{pmatrix} + \sum_{i=1}^{q} \lambda_i^h \begin{pmatrix} 0 \\ 0 \\ \nabla h_i(z^*) \end{pmatrix} - \sum_{i=1}^{m} \lambda_i^G \begin{pmatrix} -e_i \\ 0 \\ \nabla G_i(z^*) \end{pmatrix} - \sum_{i=1}^{m} \lambda_i^H \begin{pmatrix} 0 \\ -e_i \\ \nabla H_i(z^*) \end{pmatrix} - \begin{pmatrix} \xi \\ \gamma \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where e_i denotes the unit vector whose i^{th} component is equal to 1. It follows that

$0=\lambda^G+\xi,\ 0=\lambda^H+\gamma,$

$$\sum_{i=1}^{l}\tau_i\nabla f_i(z^*)+\sum_{i\in I_g}\lambda_i^g\nabla g_i(z^*)+\sum_{i=1}^{q}\lambda_i^h\nabla h_i(z^*)-\sum_{i=1}^{m}[\lambda_i^G\nabla G_i(z^*)+\lambda_i^H\nabla H_i(z^*)]=0,$$

$\lambda_{I_g}^g \geqq 0$, Since $(\xi,\gamma)\in N_\Omega(x^*,y^*)$, and

$$N_\Omega(x^*,y^*)=\begin{cases}\xi_i=0 & if\quad x_i^*>0,\\ (\xi,\gamma):\quad \gamma_i=0 & if\quad y_i^*>0,\\ either\ \xi_i<0,\gamma_i<0\ or\ \xi_i\gamma_i=0 & if\ x^*=0,\ y^*=0.\end{cases}$$

See, [3] the assertion of the theorem follows. This completes the proof.

By the Fritz John type M-stationary condition, if $\tau=(\tau_1,\dots,\tau_l)\geqq 0$ in the condition is never zero, then it can be taken as $(1,0,\dots,0)$. Hence, the following KKT type M-stationary condition follows immediately.

Theorem 3.6. (Kuhn-Tucker type M-stationary condition) Let z^* be a local efficient solution for the (MMPEC), where all functions are continuously differentiable at z^*. Suppose that (NNAMCQ) is satisfied at z^*, then z^* is M-stationary.

Proof. Since (NNAMCQ) is satisfied at z^*, $\tau=(\tau_1,\dots,\tau_l)\geqq 0$ in the Fritz John necessary condition can be $(1,0,\dots,0)$ *i.e.,* z^* is M-stationary.

This completes the proof.

In the following theorem, we extend the Theorem 2.3 of Ye [8] for multiobjective mathematical programming problems with equilibrium constraints.

Theorem 3.7. Let z^* be a feasible point of the (MMPEC) and the M-stationary conditions hold at z^*, *i.e.* there exist $\tau=(\tau_1,\dots,\tau_l)\geqq 0$ and $\lambda=(\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^{p+q+2m}$, such that

$$\sum_{i=1}^{l}\tau_i\nabla f_i(z^*)+\sum_{i\in I_g}\lambda_i^g\nabla g_i(z^*)+\sum_{i=1}^{q}\lambda_i^h\nabla h_i(z^*)-\sum_{i=1}^{m}[\lambda_i^G\nabla G_i(z^*)+\lambda_i^H\nabla H_i(z^*)]=0, \quad (3.3)$$

$$\lambda_{I_g}^g\geqq 0,\quad \lambda_\gamma^G=0,\ \lambda_\alpha^H=0,\ \forall i\in\beta \text{ either } \lambda_i^G>0,\ \lambda_i^H>0 \text{ or } \lambda_i^G\lambda_i^H=0.$$

Further, suppose that $(f,g_{I_g},h_{J^+},-h_{J^-},-G_{\alpha^-\cup\beta_H^-},-G_{\alpha^+\cup\beta_H^+\cup\beta^+},-H_{\gamma^-\cup\beta_G^-},H_{\gamma^+\cup\beta_G^+\cup\beta^+})$ are pseudoquasi-type I, then in the case when $\alpha^-\cup\gamma^-\cup\beta_G^-\cup\beta_H^-=\phi$, z^* is an efficient solution of the (MMPEC); in the case when $\beta_G^-\cup\beta_H^-=\phi$, or when z^* is an interior point relative to the set $P\cap\{z:G_i(z)=0,H_i(z)=0,i\in\beta_G^-\cup\beta_H^-\}$ *i.e.* for all feasible point *z* which is close to z^*, it holds that $G_i(z)=0,H_i(z)=0,i\in\beta_G^-\cup\beta_H^-$, z^* is a local efficient solution of the (MMPEC), where *P* denotes the set of feasible solution of the (MMPEC).

Proof. Since $(f, g_{I_g}, h_{J^+}, -h_{J^-}, -G_{\alpha^- \cup \beta_H^-}, -G_{\alpha^+ \cup \beta_H^+ \cup \beta^+}, -H_{\gamma^- \cup \beta_G^-}, H_{\gamma^+ \cup \beta_G^+ \cup \beta^+})$ is pseudoquasi-type I, it follows that

$$\langle \nabla g_i(z^*), \eta(z, z^*) \rangle \leqq 0, \forall i \in I_g. \qquad (3.4)$$

Similarly, we have

$$\langle \nabla h_i(z^*), \eta(z, z^*) \rangle \leqq 0, \forall i \in J^+, \qquad (3.5)$$

$$-\langle \nabla h_i(z^*), \eta(z, z^*) \rangle \leqq 0, \forall i \in J^+. \qquad (3.6)$$

By the definition of pseudoquasi-type I for $G_i(\forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$ and $H_i(\forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, we have

$$\langle \nabla G_i(z^*), \eta(z, z^*) \rangle \geqq 0, \forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+, \quad (3.7)$$

$$\langle \nabla H_i(z^*), \eta(z, z^*) \rangle \geqq 0, \forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+. \quad (3.8)$$

If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, multiplying (3.4) to (3.8), by $\lambda_{I_g}^g \geqq 0 \ (i \in I_g)$, $\lambda_i^h > 0 \ (i \in J^+), -\lambda_i^h > 0 \ (i \in J^-)$, $\lambda_i^G > 0 \ (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $\lambda_i^H > 0 \ (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, respectively and adding, we get

$$\left\langle \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)], \eta(z, z^*) \right\rangle \leqq 0.$$

Using (3.3) above inequality implies that

$$\left\langle \sum_{i=1}^{l} \tau_i \nabla f_i(z^*), \eta(z, z^*) \right\rangle \geqq 0.$$

Thus the definition of pseudoquasi-type I at z^*, we get $\sum_{i=1}^{l} \tau_i f_i(z) \geqq \sum_{i=1}^{l} \tau_i f_i(z^*)$ for all feasible point z and hence z^* is an efficient solution for the (MMPEC) if $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$.

Suppose that $\alpha^- \cup \gamma^- \neq \phi$ and $\beta_G^- \cup \beta_H^- = \phi$. For any $i \in \alpha$, since $H_i(z^*) > 0, H_i(z) > 0$ for sufficiently close to z^*, we have

$$G_i(z) = G_i(z^*), \forall i \in \alpha.$$

By definition of pseudoquasi-type I of $G_i(\forall i \in \alpha^-)$ at z^* it follows that for z sufficiently to z^*,

$$-\langle \nabla G_i(z^*), \eta(z, z^*) \rangle \geqq 0, \forall i \in \alpha^-. \qquad (3.9)$$

Similarly, we have for z sufficiently close to z^*,

$$-\langle \nabla H_i(z^*), \eta(z,z^*)\rangle \geqq 0, \forall i \in \gamma^-. \qquad (3.10)$$

Multiplying (3.4) to (3.10) by $\lambda_{I_g}^g \geqq 0(i \in I_g)$, $\lambda_i^h > 0(i \in J^+), -\lambda_i^h > 0(i \in J^-)$, $\lambda_i^G > 0(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $\lambda_i^H > 0(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+), -\lambda_i^G > 0(i \in \alpha^-), -\lambda_i^H > 0(i \in \gamma^-)$, respectively and adding, we have that for z sufficiently close to z^*,

$$\left\langle \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)], \eta(z,z^*) \right\rangle \leqq 0.$$

Using (3.3), the above inequality implies that for z sufficiently close to z^*,

$$\left\langle \sum_{i=1}^{l} \tau_i \nabla f_i(z^*), \eta(z,z^*) \right\rangle \geqq 0.$$

Then, by the definition of pseudoquasi-type I at z^*, we get

$$\sum_{i=1}^{l} \tau_i f_i(z) \geqq \sum_{i=1}^{l} \tau_i f_i(z^*).$$

For z sufficiently close to z^*, that is z^* is local solution for the (MMPEC) if $\alpha^- \cup \gamma^- = \phi$, and $\beta_G^- \cup \beta_H^- = \phi$.

Now suppose that z^* is an interior point relative to the set $P \cap \{z : G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \cup \beta_H^-\}$, then for any feasible point z sufficiently close to z^*, it holds that $G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \cup \beta_H^-$, and hence by definition of pseudoquasi-type I of $G_i(i \in \beta_H^-)$ and $H_i(i \in \beta_G^-)$,

$$-\langle \nabla G_i(z^*), \eta(z,z^*)\rangle \geqq 0, \forall i \in \beta_H^-, \qquad (3.11)$$

$$-\langle \nabla H_i(z^*), \eta(z,z^*)\rangle \geqq 0, \forall i \in \beta_G^-. \qquad (3.12)$$

Multiplying (3.4) to (3.12) by $\lambda_{I_g}^g \geqq 0(i \in I_g)$, $\lambda_i^h > 0(i \in J^+), -\lambda_i^h > 0(i \in J^-), \lambda_i^G > 0(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+), -\lambda_i^G > 0(i \in \alpha^- \cup \beta_H^-)$, $-\lambda_i^H > 0(i \in \gamma^- \cup \beta_G^-)$, respectively and adding, we have that for z sufficiently close to z^*,

$$\left\langle \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)], \eta(z,z^*) \right\rangle \leqq 0.$$

Using (3.3), the above inequality implies that for z sufficiently close to z^*,

$$\left\langle \sum_{i=1}^{l} \tau_i \nabla f_i(z^*), \eta(z, z^*) \right\rangle \geqq 0.$$

Then, by the definition of pseudoquasi-type I at z^*, we get

$$\sum_{i=1}^{l} \tau_i f_i(z) \geqq \sum_{i=1}^{l} \tau_i f_i(z^*) .$$

For z sufficiently close to z^*, that is z^* is local solution for the (MMPEC) if z^* is an interior point relative to the set $P \bigcap \{z : G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \bigcup \beta_H^-\}$.

This completes the proof.

Now, we give an example to illustrate Theorem 3.7.

Example 3.8. Consider the following MMPEC in $\mathbb{R}^2$.

$$\min \quad f(z_1, z_2) = (z_1, z_2)$$

$$\text{subject to} \quad G(z_1, z_2) = z_1 \geqq 0,$$

$$H(z_1, z_2) = z_2 \geqq 0,$$

$$G(z_1, z_2) H(z_1, z_2) = z_1 z_2 = 0, \quad \forall\, z_1, z_2 \in \mathbb{R}.$$

Let $f_1(z_1, z_2) = z_1$ and $f_2(z_1, z_2) = z_2$. The feasible region of MMPEC is $P = \{(z_1, z_2) \in \mathbb{R}^2$ such that either $z_1 = 0$ and $z_2 \geqq 0$ or $z_2 = 0$ and $z_1 \geqq 0\}$. If we take point $z^* = (0,0)$ in feasible region then index set $\gamma(0,0)$ are empty set but $\beta(0,0) = \{1\}$. Also $\lambda_1 \nabla f_1(0,0) + \lambda_2 \nabla f_2(0,0) - \mu \nabla G(0,0) - \upsilon \nabla H(0,0) = 0$ and either $\mu\upsilon = 0$ or $\mu > 0$ and $\upsilon > 0$ then, $\lambda_1 - \mu = 0, \lambda_2 - \upsilon = 0$ and either $\mu\upsilon = 0$ or $\mu > 0$ and $\upsilon > 0$. If we take $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2},$ then $\mu = \frac{1}{2}, \upsilon = \frac{1}{2}$ such that MMPEC M-stationary conditions hold. Therefore, by Theorem 3.7, $z^* = (0,0)$ is an efficient solution of MMPEC.

4. Duality

We formulate Wolfe type and Mond-Weir type duals and generalize the duality results using type I and pseudoquasi-type I assumptions. We propose the Wolfe type dual for the multiobjective mathematical programming problems with equilibrium constraints the (MMPEC) using generalized convexity.

$$\text{(WDMMPE} \quad \max \quad f(u) + \left[\sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) - \sum_{i=1}^{m} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)] \right] e,$$

where $e = (1,1,\ldots,1) \in \mathbb{R}^l$,

subject to

$$\sum_{i=1}^{l}\tau_i\nabla f_i(u)+\sum_{i\in I_g}\lambda_i^g\nabla g_i(u)+\sum_{i=1}^{q}\lambda_i^h\nabla h_i(u)-\sum_{i=1}^{m}[\lambda_i^G\nabla G_i(u)+\lambda_i^H\nabla H_i(u)]=0,$$

$$\lambda_{I_g}^g \geqq 0,\ \lambda_{\bar{\gamma}}^G=0, \lambda_{\bar{\gamma}}^H=0, \forall i\in\bar{\beta}\ \text{either}\ \lambda_i^g>0, \lambda_i^G>0,\ \text{or}\ \ \lambda_i^H\lambda_i^G=0,$$

where, $\lambda=(\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^{p+q+2m}$, $\tau=(\tau_1,\dots,\tau_l)\geqq 0$ and $\sum_{i=1}^{l}\tau_i=1$.

Also,

$$\bar{\alpha}=\bar{\alpha}(u)=\{i:G_i(u)=0, H_i(u)\neq 0\},$$

$$\bar{\beta}=\bar{\beta}(u)=\{i:G_i(u)=0, H_i(u)=0\},$$

$$\bar{\gamma}=\bar{\gamma}(u)=\{i:G_i(u)\neq 0, H_i(u)=0\},$$

where $i\in\{1,2,\dots,m\}$.

Theorem 4.1. (Weak Duality) Let z be a feasible for the (MMPEC) and (u,τ,λ) be feasible for the (WDMMPEC) where $\lambda=(\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^{p+q+2m}$ and $\tau\in\mathbb{R}^l$. Let

$$J^+=\{i:\lambda_i^h>0\},\ J^-=\{i:\lambda_i^h<0\},$$

$$\bar{\beta}^+=\{i\in\bar{\beta}:\lambda_i^G>0, \lambda_i^H>0\},$$

$$\bar{\beta}_G^+=\{i\in\bar{\beta}:\lambda_i^G=0, \lambda_i^H>0\},\ \bar{\beta}_G^-=\{i\in\bar{\beta}:\lambda_i^G=0, \lambda_i^H<0\},$$

$$\bar{\beta}_H^+=\{i\in\bar{\beta}:\lambda_i^H=0, \lambda_i^G>0\},\ \bar{\beta}_H^-=\{i\in\bar{\beta}:\lambda_i^H=0, \lambda_i^G<0\},$$

$$\bar{\alpha}^+=\{i\in\bar{\alpha}:\lambda_i^G>0\},\ \bar{\alpha}^-=\{i\in\bar{\alpha}:\lambda_i^G<0\},$$

$$\bar{\gamma}^+=\{i\in\bar{\gamma}:\lambda^H>0\},\ \bar{\gamma}^-=\{i\in\bar{\gamma}:\lambda^H<0\}.$$

Also, suppose that $(f, g_{I_g}, h_{J^+}, -h_{J^-}, -G_{\bar{\alpha}^-\cup\bar{\beta}_H^-}, -G_{\bar{\alpha}^+\cup\bar{\beta}_H^+\cup\bar{\beta}^+}, -H_{\bar{\gamma}^-\cup\bar{\beta}_G^-}, -H_{\bar{\gamma}^+\cup\bar{\beta}_G^+\cup\bar{\beta}^+})$ are semi-strictly-type I at u with respect to a common kernel η and if $\bar{\alpha}^-\cup\bar{\gamma}^-\cup\bar{\beta}_G^-\cup\bar{\beta}_H^-=\phi$. Then, for any z feasible for the (MMPEC), we have

$$f(z)\not\leq f(u)+\left[\sum_{i\in I_g}\lambda_i^g g_i(u)+\sum_{i=1}^{q}\lambda_i^h h_i(u)\ -\sum_{i=1}^{m}[\lambda_i^G G_i(u)+\lambda_i^H H_i(u)]\right]e.$$

Proof. Let

$$f(z)\leq f(u)+\left[\sum_{i\in I_g}\lambda_i^g g_i(u)+\sum_{i=1}^{q}\lambda_i^h h_i(u)\ -\sum_{i=1}^{m}[\lambda_i^G G_i(u)+\lambda_i^H H_i(u)]\right]e,$$

then there exist index p such that

$$f_p(z)<f_p(u)+\left[\sum_{i\in I_g}\lambda_i^g g_i(u)+\sum_{i=1}^{q}\lambda_i^h h_i(u)\ -\sum_{i=1}^{m}[\lambda_i^G G_i(u)+\lambda_i^H H_i(u)]\right],$$

$$f_i(z) \leqq f_i(u)+\left[\sum_{i\in I_g}\lambda_i^g g_i(u)+\sum_{i=1}^{q}\lambda_i^h h_i(u)\ -\sum_{i=1}^{m}[\lambda_i^G G_i(u)+\lambda_i^H H_i(u)]\right],\ \forall i\neq p.$$

Since by hypothesis, the above inequality gives

$$\left\langle \sum_{i=1}^{l}\tau_i\nabla f_i(u)+\sum_{i\in I_g}\lambda_i^g\nabla g_i(u)+\sum_{i=1}^{q}\lambda_i^h\nabla h_i(u)-\sum_{i=1}^{m}[\lambda_i^G\nabla G_i(u)+\lambda_i^H\nabla H_i(u)],\eta(z,u)\right\rangle<0.$$

Then,

$$\sum_{i=1}^{l}\tau_i\nabla f_i(u)+\sum_{i\in I_g}\lambda_i^g\nabla g_i(u)+\sum_{i=1}^{q}\lambda_i^h\nabla h_i(u)-\sum_{i=1}^{m}[\lambda_i^G\nabla G_i(u)+\lambda_i^H\nabla H_i(u)]<0.$$

We obtained a contradiction to the duality constraints in the feasibility of the (u,τ,λ) for the (WDMMPEC). Hence,

$$f(z)\not\leq f(u)+\left[\sum_{i\in I_g}\lambda_i^g g_i(u)+\sum_{i=1}^{q}\lambda_i^h h_i(u)\ -\sum_{i=1}^{m}[\lambda_i^G G_i(u)+\lambda_i^H H_i(u)]\right]e.$$

This completes the proof.

Theorem 4.2. (Strong Duality) Let z^* be an efficient solution for the (MMPEC) and satisfies the (NNAMCQ) at z^*. Let $(f,g_{I_g},h_{J^+},-h_{J^-},-G_{\bar{\alpha}^-\cup\bar{\beta}_H^-},-G_{\bar{\alpha}^+\cup\bar{\beta}_H^+\cup\bar{\beta}^+},-H_{\bar{\gamma}^-\cup\bar{\beta}_G^-},-H_{\bar{\gamma}^+\cup\bar{\beta}_G^+\cup\bar{\beta}^+})$ be semi-strictly-type I at z^* with respect to the common kernel η and $\bar{\alpha}^-\cup\bar{\gamma}^-\cup\bar{\beta}_G^-\cup\bar{\beta}_H^-=\phi$, where $J^+, J^-, \bar{\beta}^+, \bar{\beta}_G^+, \bar{\beta}_G^-, \bar{\beta}_H^+, \bar{\beta}_H^-, \bar{\alpha}^+, \bar{\gamma}^+$ and $\bar{\gamma}^-$ are defined in Theorem 4.1. Then, $\exists\,\bar{\lambda}=(\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)\in\mathbb{R}^{p+q+2m}$ and $\bar{\tau}\in\mathbb{R}^l$ such that $(z^*,\bar{\tau},\bar{\gamma})$ is an efficient solution for the (WDMMPEC) and respective objective values are equal.

Proof. As z^* be an efficient solution for the (MMPEC) and satisfies the (NNAMCQ) at z^*. Hence, $\exists\,\bar{\lambda}=(\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)\in\mathbb{R}^{p+q+2m}$ and $\bar{\tau}_i\geqq 0\ i\in\{1,2,\dots,l\}$ such that the M-stationary conditions for the (MMPEC) satisfied

$$\sum_{i=1}^{l}\bar{\tau}_i\nabla f_i(z^*)+\sum_{i\in I_g}\bar{\lambda}_i^g\nabla g_i(z^*)+\sum_{i=1}^{q}\bar{\lambda}_i^h\nabla h_i(z^*)-\sum_{i=1}^{m}[\bar{\lambda}_i^G\nabla G_i(z^*)+\bar{\lambda}_i^H\nabla H_i(z^*)]=0,$$

$$\bar{\lambda}_{I_g}^g\geqq 0,\ \bar{\lambda}_\gamma^G=0,\bar{\lambda}_\alpha^H=0,\forall i\in\beta \text{ either } \bar{\lambda}_i^G>0,\bar{\lambda}_i^H>0,\ \text{or}\ \ \bar{\lambda}_i^G\bar{\lambda}_i^H=0.$$

Therefore, $(z^*,\bar{\tau},\bar{\gamma})$ is feasible for the (WDMMPEC) by Theorem 4.1, we have

$$f(z^*)\not\leq f(u)+\left[\sum_{i\in I_g}\lambda_i^g g_i(u)+\sum_{i=1}^{q}\lambda_i^h h_i(u)\ -\sum_{i=1}^{m}[\lambda_i^G G_i(u)+\lambda_i^H H_i(u)]\right]e, \qquad (4.1)$$

for any feasible solution (z,τ,γ) for the (WDMMPEC).

Also, from the feasibility condition of the (MMPEC) and the (WDMMPEC) that is for $i \in I_g(z^*), g_i(z^*) = 0$, also $h_i(z^*) = 0, G_i(z^*) = 0, \forall i \in \alpha \cup \beta$ and $H_i(z^*) = 0, \forall i \in \beta \cup \gamma$, then we have

$$f(z^*) = f(z^*) + \left[\sum_{i \in I_g} \bar{\lambda}_i^g g_i(z^*) + \sum_{i=1}^{q} \bar{\lambda}_i^h h_i(z^*) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(z^*) + \bar{\lambda}_i^H H_i(z^*)] \right] e. \tag{4.2}$$

Using (4.1) and (4.2), we have

$$\begin{aligned} f(z^*) &= f(z^*) + \left[\sum_{i \in I_g} \bar{\lambda}_i^g g_i(z^*) + \sum_{i=1}^{q} \bar{\lambda}_i^h h_i(z^*) - \sum_{i=1}^{m} [\bar{\lambda}_i^G G_i(z^*) + \bar{\lambda}_i^H H_i(z^*)] \right] e \\ &\nleq f(u) + \left[\sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) - \sum_{i=1}^{m} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)] \right] e. \end{aligned}$$

Hence $(z^*, \bar{\tau}, \bar{\gamma})$ is an efficient solution for the (WDMMPEC) and the respective values are equal for the suitable choice of $(\bar{\tau}, \bar{\gamma})$.

We propose the following Mond-Weir type dual of the (MMPEC).

(MWDMMPEC) $\quad\quad \max \quad f(u)$

subject to

$$\sum_{i=1}^{l} \tau_i \nabla f_i(u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m} \left[\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u) \right] = 0, \tag{4.3}$$

$$\sum_{i \in I_g} \lambda_i^g g_i(u) \geqq 0, \quad \sum_{i=1}^{q} \lambda_i^h h_i(u) \geqq 0,$$

$$\sum_{i=1}^{m} \lambda_i^G G_i(u) \leqq 0, \quad \sum_{i=1}^{m} \lambda_i^H H_i(u) \leqq 0,$$

$$\lambda_i^g \geqq 0, \ \lambda_{\gamma}^G = 0, \lambda_{\gamma}^H = 0, \forall i \in \bar{\beta} \ \text{ either } \lambda_i^g > 0, \lambda_i^G > 0, \ \text{ or } \ \lambda_i^H \lambda_i^G = 0,$$

where, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2m}, \tau_i \geqq 0 \ i \in \{1,2,\dots,l\}$ and $\sum_{i=1}^{l} \tau_i = 1$.

Theorem 4.3. (Weak Duality) Let z be a feasible for the (MMPEC) and (u, τ, λ) be feasible for the (MWDMMPEC) where $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2m}$ and $\tau \in \mathbb{R}^l$. Let

$$J^+ = \{i : \lambda_i^h > 0\}, \ J^- = \{i : \lambda_i^h < 0\},$$

$$\bar{\beta}^+ = \{i \in \bar{\beta} : \lambda_i^G > 0, \lambda_i^H > 0\},$$

$$\bar{\beta}_G^+ = \{i \in \bar{\beta} : \lambda_i^G = 0, \lambda_i^H > 0\}, \ \bar{\beta}_G^- = \{i \in \bar{\beta} : \lambda_i^G = 0, \lambda_i^H < 0\},$$

$$\bar{\beta}_H^+ = \{i \in \bar{\beta} : \lambda_i^H = 0, \lambda_i^G > 0\},\ \bar{\beta}_H^- = \{i \in \bar{\beta} : \lambda_i^H = 0, \lambda_i^G < 0\},$$

$$\bar{\alpha}^+ = \{i \in \bar{\alpha} : \lambda_i^G > 0\},\ \bar{\alpha}^- = \{i \in \bar{\alpha} : \lambda_i^G < 0\},$$

$$\bar{\gamma}^+ = \{i \in \bar{\gamma} : \lambda^H > 0\},\ \bar{\gamma}^- = \{i \in \bar{\gamma} : \lambda^H < 0\}.$$

Also, suppose that $(f, g_{I_g}, h_{J^+}, -h_{J^-}, -G_{\bar{\alpha}^- \cup \bar{\beta}_H^-}, -G_{\bar{\alpha}^+ \cup \bar{\beta}_H^+ \cup \bar{\beta}^+}, -H_{\bar{\gamma}^- \cup \bar{\beta}_G^-}, -H_{\bar{\gamma}^+ \cup \bar{\beta}_G^+ \cup \bar{\beta}^+})$ are pseudoquasi-type I at u with respect to a common kernel η with $\tau_i > 0, \forall i \in \{1, 2, ..., l\}$ and if $\bar{\alpha}^- \cup \bar{\gamma}^- \cup \bar{\beta}_G^- \cup \bar{\beta}_H^- = \phi$.

Then for any z feasible for the (MMPEC), we have

$$f(z) \nleq f(u).$$

Proof. Let $f(z) \leq f(u)$. Then, there exist some p such that

$$f_p(z) < f_p(u),$$

$$f_i(z) \leqq f_i(u), \forall i \neq p.$$

Since $\tau_i > 0,\ i = 1, 2, ..., l$

$$\sum_{i=1}^{l} \tau_i f_i(z) < \sum_{i=1}^{l} \tau_i f_i(u) \Rightarrow \sum_{i=1}^{l} \tau_i [\nabla f_i(u)]^t \eta(z, u) < 0 \qquad (4.4)$$

and

$$-\sum_{i \in I_g} \lambda_i^g g_i(u) \leqq 0 \Rightarrow \sum_{i \in I_g} \lambda_i^g [g_i(u)]^t \eta(z, u) \leqq 0, \qquad (4.5)$$

$$-\sum_{i=1}^{q} \lambda_i^h h_i(u) \leqq 0 \Rightarrow \sum_{i=1}^{q} \lambda_i^h [\nabla h_i(u)]^t \eta(z, u) \leqq 0, \qquad (4.6)$$

$$\sum_{i=1}^{m} \lambda_i^G G_i(u) \geqq 0 \Rightarrow \sum_{i=1}^{m} \lambda_i^G [\nabla G_i(u)]^t \eta(z, u) \geqq 0, \qquad (4.7)$$

$$\sum_{i=1}^{m} \lambda_i^H H_i(u) \geqq 0 \Rightarrow \sum_{i=1}^{m} \lambda_i^H [\nabla H_i(u)]^t \eta(z, u) \geqq 0. \qquad (4.8)$$

By definition of pseodoquasi-type I, adding (4.4) to (4.8), we get

$$\sum_{i=1}^{l} \tau_i \nabla f_i(u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)] < 0,$$

we obtained a contradiction then,

$$f(z) \nleq f(u).$$

This completes the proof.

Theorem 4.4. (Strong Duality) Let z^* be an efficient solution for the (MMPEC) and satisfies the No Nonzero Abnormal Multiplier Constraints Qualification (NNAMCQ) at z^*. Let $(f, g_{I_g}, h_{J^+}, -h_{J^-}, -G_{\bar{\alpha}^- \cup \bar{\beta}_H^-}, -G_{\bar{\alpha}^+ \cup \bar{\beta}_H^+ \cup \bar{\beta}^+}, -H_{\bar{\gamma}^- \cup \bar{\beta}_G^-}, -H_{\bar{\gamma}^+ \cup \bar{\beta}_G^+ \cup \bar{\beta}^+})$ be pseudoquasi-type I at z^* with respect to common kernel η and $\bar{\alpha}^- \cup \bar{\gamma}^- \cup \bar{\beta}_G^- \cup \bar{\beta}_H^- = \phi$, where J^+, J^-, $\bar{\beta}^+$, $\bar{\beta}_G^+$, $\bar{\beta}_G^-$, $\bar{\beta}_H^+$, $\bar{\beta}_H^-$, $\bar{\alpha}^+$, $\bar{\gamma}^+$ and $\bar{\gamma}^-$ are defined in Theorem 4.3. Then, $\exists\, \bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{p+q+2m}$ and $\bar{\tau} \in \mathbb{R}^l$, such that $(z^*, \bar{\tau}, \bar{\gamma})$ is an efficient solution for the (MWDMMPEC) and respective values are equal.

Proof. As z^* be an efficient solution for the (MMPEC) are satisfied the (NNAMCQ) at z^*. Hence, $\exists\, \bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{p+q+2m}$ and$\tau_i \geqq 0$ $i \in \{1,2,\dots,l\}$such that the M-stationary conditions for the (MMPEC) satisfied i.e.,

$$\sum_{i=1}^{l} \bar{\tau}_i \nabla f_i(z^*) + \sum_{i \in I_g} \bar{\lambda}_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \bar{\lambda}_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\bar{\lambda}_i^G \nabla G_i(z^*) + \bar{\lambda}_i^H \nabla H_i(z^*)] = 0,$$

$$\bar{\lambda}_{I_g}^g \geqq 0, \bar{\lambda}_\gamma^G = 0,\ \bar{\lambda}_\alpha^H = 0,\ \forall i \in \beta \text{ either } \bar{\lambda}_i^G > 0,\ \bar{\lambda}_i^H > 0, \text{ or } \bar{\lambda}_i^G \bar{\lambda}_i^H = 0.$$

Also from the feasibility conditions of the (MMPEC) and the (WDMMPEC) that is for $i \in I_g(z^*)$, $g_i(z^*) = 0$, also $h_i(z^*) = 0$, $G_i(z^*) = 0$, $\forall i \in \alpha \cup \beta$ and $H_i(z^*) = 0$, $\forall i \in \beta \cup \gamma$, then we have

$$\sum_{i \in I_g} \bar{\lambda}_i^g g_i(z^*) = 0, \quad \sum_{i=1}^{q} \bar{\lambda}_i^h h_i(z^*) = 0,$$

$$\sum_{i=1}^{m} \bar{\lambda}_i^G G_i(z^*) = 0, \quad \sum_{i=1}^{m} \bar{\lambda}_i^H H_i(z^*) = 0.$$

Since z^* is an efficient solution for the (MWDMMPEC) and respective objective values are equal for suitable choice of $(z^*, \bar{\tau}, \bar{\lambda})$ is feasible for the (MWDMMPEC). By weak duality

$$f(z^*) \nleq f(u).$$

Thus, $(z^*, \bar{\tau}, \bar{\lambda})$ is an efficient solution for the (MWDMMPEC) and respective objective values are equal for suitable choice of $(\bar{\tau}, \bar{\lambda})$. This completes the proof.

Conclusions

We studied multiobjective mathematical programming problems with equilibrium constraints. We established necessary and sufficient optimality conditions for multiobjective mathematical programming problems with equilibrium constraints using generalized convexity. Also, we proposed Wolfe type dual and Mond-Weir type dual models and established weak and strong duality results for multiobjective mathematical programming problems with equilibrium constraints using generalized convexity.

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